Scale Analysis

Purpose and Methodology

by Paul Poletti©1997

Originally read at the Accademia Bartolomeo Cristofori
Florence, Italy, December 1997

The author wishes to thank Stefano Fiuzzi and the staff of ABC for having made this most enjoyable and rewarding conference possible

This is version 20/12/02

About 2500 years, Pythagoras of Samos founded a school of philosophers and mathematicians whose ideas were to profoundly affect the intellectual history of Western Europe. Pythagoreans believed that reality could be explained through an underlying set of numerical relations, a generalization they arrived at through observations in music, mathematics, and astronomy. Not only did their theorems establish many of the basic tenants in these disciplines, but the very philosophical foundation of modern science is based on their fundamental doctrine, the belief that behind our reality lies a structure of rational order which can be discovered through empirical investigation.

Today, we are still traveling down the path begun by Pythagoras and his followers, but the basic idea of a sublime order in the universe is under siege. As we look deeper and deeper into the structure of our reality, we find only more ambiguities and contradictions. Rallying to the challenge, modern mathematics has developed a new approach, Chaos Theory, to try to rationalize and quantify precisely those aspects of things which behave irrationally and seem unquantifiable.

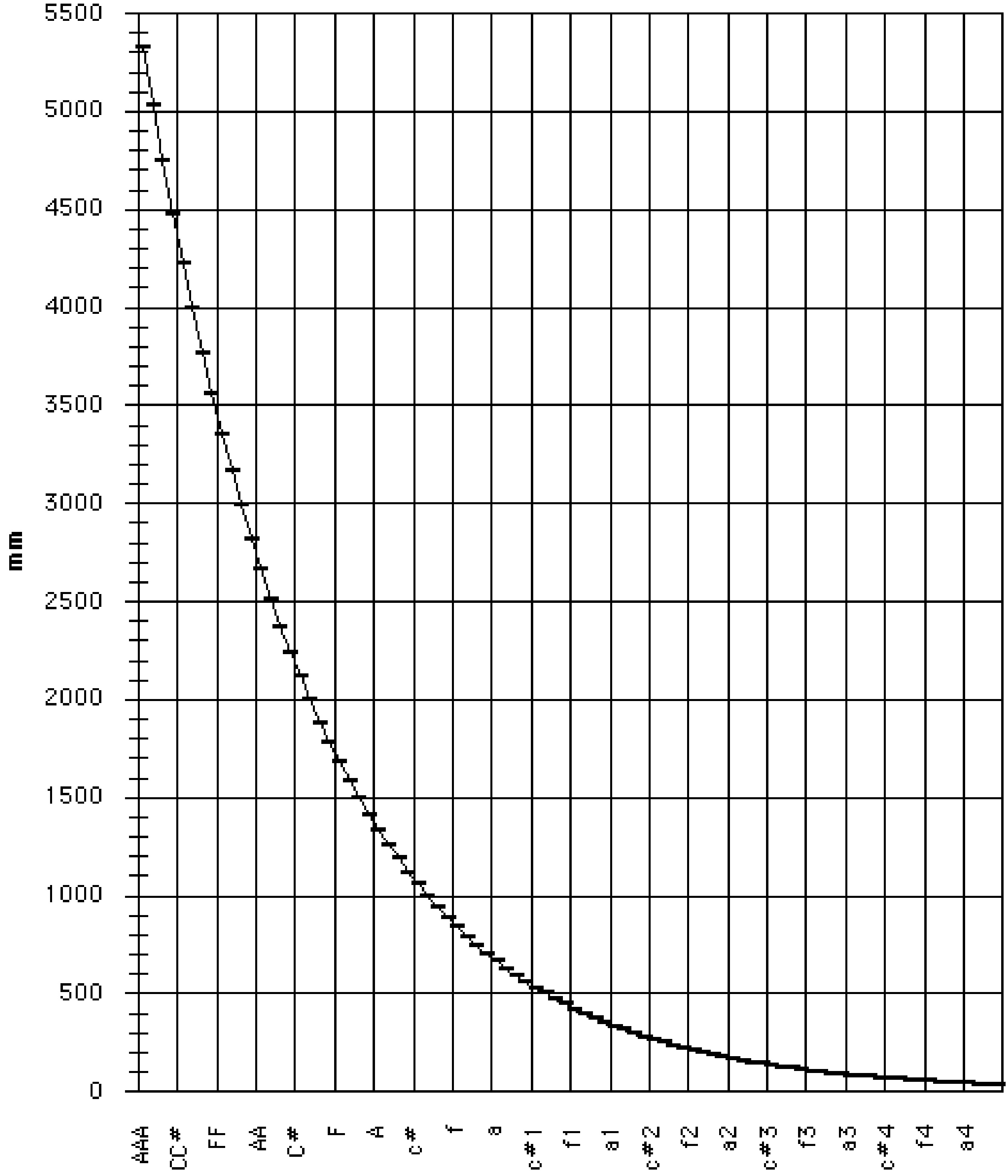
Luckily, our work as organologists does not function at these levels. There is, however, one aspect of our investigation which does seem rather chaotic: the topic of scaling and stringing. The record of extant instruments and documents provides us with a baffling variety of string lengths, pitch levels, wire gauge systems and various mismatched and sometimes conflicting indicators of wire strength. Furthermore, there is no common approach to examining these topics; those interested must become familiar with several different ways of describing scale length and shape, and a plethora of measurement units: c^2-equivalents, Megapascals, kilograms-force per millimeter squared, Newtons, percentage of yield strength, etc. Any real understanding of scaling is almost impossible without the help of a personal computer, or at the very least, a scientific calculator. One example is more than enough to illustrate the complex manner with which the topic is commonly approached; in a soon to appear article on Broadwood, Cavallo, and the appearance of divided bridges in English pianos, John Koster wishes to demonstrate that brass wire on a divided-bridge Broadwood was under considerably less tension than in earlier single-bridge instruments, a tension level well below that at which the wire would break. To prove this, he begins with a reported strength for 18th-century brass wire of 70kgf/mm^2 for a diameter of 0.50mm. He then states that the highest bass-bridge notes on a Broadwood piano have a c^2 equivalent of 215mm, and from this he derives that this wire would be 4 to 5 semitones below the pitch at which it would break. Furthermore, he compares this with a contemporary Stodart instrument with a single bridge, converting its G# string of 1233mm to a c^2 equivalent of 245mm. End of proof. Those of us well-versed in the topic can follow the basic drift, but for the rest of you, I can well imagine that it's about as clear as mud. Even for the cognoscenti, though, the proof given is not a real proof, for the conclusion, which is in fact correct, only becomes a certainty after considerable further mathematical analysis.

Luckily for us, in regards to scaling, the ideas of the Pythagoreans area alive and well, both literally and figuratively. The purpose of my lecture is twofold: first, I hope to demonstrate that scaling analysis *can* be viewed using a system which is simple, all-encompassing, and perhaps even historical; and furthermore, that there is in fact one simple proportion which lies behind all of the phenomena associated with scaling, a proportion which can also be used to great advantage in analysis. Secondly, by drawing upon examples and problems in recent organology, I want to demonstrate how such a system can make our work both easier and more comprehensible for our readers, and also show how it often can produce alternative conclusions or bring to light important relationships obscured by other means of analysis. In so doing, I will make reference to and use of the work of my good colleagues, some of whom are present today. To you, I offer in advance my apologies for putting you on the spot. Furthermore, in an attempt to make the topic clear to everyone, I will also need to go over some very basic ground, and I ask those with more advanced understanding to please bear with me.

Before launching into the theory, though, I see by the title of my paper that I'm also supposed to say something about *why* we study scaling. Quite frankly, some of us probably do it because we enjoy mucking about with numbers. But, on a more practical level, the study of scaling has some very important applications. Along with decorational and structural characteristics, scale analysis can also be a very useful tool to help place anonymous instruments within the various schools of particular regions, times, and builders, or to identify fraudulent instruments. For modern instrument makers and restorers, a firm understanding of scaling is needed to reconstruct lost stringing schemes, or to interpret the often confusing evidence of original gauge markings. Scaling is one of the most critical design elements, for a mistake in string length, weight, or strength can produce unsatisfactory outcomes, such as collapsing instruments or breaking strings. Both scaling and stringing play a major role in the final tone of the instrument, and they also have a purely functional role in terms of touch and mechanical reliability.

Now that that's out of the way, let's get down to some serious mucking-about with numbers. Pitch is inversely proportional to string length. This basic fact was first described by the Pythagoreans, but it has probably been practical knowledge among musicians and instrument makers since Day One. Since our western musical system is based on pure octaves which are more or less equally divided by 12 tones, a basic theoretical scale shape has octave strings that are halved or doubled, and adjacent notes which are related by the 12th root of 2.

Such a curve is called a Pythagorean curve, a shape which I'm sure is familiar to all of you:



A Pythagorean curve, of course, can be constructed starting on any note or at any length. Different types of wire and different pitch levels will necessitate different absolute lengths. Modern organologists often cite the length of the note c^2 to describe the overall length of a given scale. While this is a convenient method for roughly comparing scales, the c^2 idea is often used for types of analysis for which it is inherently unsuited, producing results which are misleading or inaccurate, as we shall see.

While the basic proportion between string length and pitch has long been known, it was the work of others, including Galileo, Coulomb, and Mersenne, which described all of the relevant factors. In 1713, the English mathematician Brook Taylor published a formula for calculating string tension. Taylor's formula can be arranged as such:

(Eq.1)

$$T=(l*f*d)^2*k$$

were **T** is tension (kilograms) **l** is length (millimeters), **f** is frequency (Hertz), **d** is diameter (millimeters), and **k** is a constant involving the density of the material and the acceleration of gravity.[1]

In scaling analysis, however, we are often not so concerned with absolutes, but with relative considerations. What happens when we string a given note heavier, for example. Or what happens when the scale becomes much shorter? For this sort of problem, Taylor's formula becomes much more useful if we substitute the absolute values with proportional representations, just as discussion of temperament becomes much easier if we use the proportional unit of cents instead of the absolute unit of Hz. What shall our unit be? In any enquiry, it is wise to select a manner of description which makes the underlying relationships clearly evident, and it also helps if we can find a unit which has some relation to the real-world application of the subject at hand. Since the basic purpose of constructing an instrument scale is to produce musical tones, a logical proportional unit would be the semitone. This is a ratio of 1:1.059, or about 6%. The basic formula for semitones is such:

(Eq.2)

for any two values X_1 and X_2

$$\text{semitones} = \log(X_1/X_2)/\log(2)*12$$

Notice that I say for <u>any</u> two values. It is critical to remember that I am using the word semitone to simply mean a unit of proportionality, and as such, can be used to describe the difference between any two numbers, regardless of what they represent. Now I know that those who are not mathematically inclined may panic at the mere appearance of the word "log" in an equation . . . it conjures up memories of incomprehensible concepts, such as the idea that 10 multiplied by itself zero number of times somehow equals 1. Rest assured, however, that using one of the cheap and simple calculators with a log key which are readily available nowadays, this calculation can be performed in about five seconds. The

[1] The constant is expressed as (m x Pi)/(9.81 x 10^{12}), where m = density (mass) in kg/m^3.

semitone formula can also be turned around in order to calculate the change in a value by a number of semitones:

(Eq.3)

given X1 and difference in semitones

$$X_2 = X_1 * 10^{(semitones/12*\log(2))}$$

Using semitones as a basic unit, Taylor's formula can now be expressed so:

(Eq. 4)

$$\Delta T=((\Delta l+\Delta p+\Delta d)*2)+\Delta k$$

substituting the letter p from "pitch" for frequency (the Greek letter Δ means "a change in the value of"). There are two distinct advantages to using proportional units: first, the complications of multiplication and the square function are replaced by simple addition and doubling; second, any factor which does not change can simply be left out of the equation. For example, since the acceleration of gravity never changes, we can leave it out, and the usual constant k can be replaced with the density of a material alone. Normally, this is represented by the Greek letter rho, but since it looks too much like a "p", I'll use "w" for weight instead.

Using such a formula, we can easily see why string tension is constant on a Pythagorean curve if one type and size of wire is used throughout. Since d and w do not change, we ignore them. The changes in length are always exactly balanced by changes in pitch; moving downward one note, for example, we get:

(Eq. 5)

$$\Delta T=(l+1)+(p-1)$$

The plus 1 in l is exactly balanced by the minus 1 in p, and the value of ΔT is zero.

But in real instruments string diameter is NOT constant: strings in the bass are always somewhat thicker than in the treble, even on instruments which are mostly Pythagorean throughout. Using a "semitone" version of Taylor's formula, any change in tension is incredibly fast and easy to calculate. Since nothing else changes, we ignore everything except diameter. Thus:

(Eq.6)

$$\Delta T=\Delta d*2$$

All we need is three numbers and one simple calculation. For example, suppose we know the tension of a given string with 0.90mm wire is 30kg; what would the tension be if we replaced the string 0.80mm wire? The difference in diameter is exactly -2 semitones, so the difference in tension would be -4 semitones. While it may seem bizarre to talk about tension in semitones, if we just plug our original tension figure into the reverse semitone formula, we get the value of 23.7kg.

This is exactly the answer we would get if we went through all the work of actually calculating the tension using the absolute values and Taylor's original formula. It is also the decrease in tension we would get if we tuned the first string down by 2 semitones in pitch, so describing tension in semitones is really not so strange after all: we need only remember that any "semitone" change in tension is twice the semitone change in pitch, length, or diameter, or the sum of changes in two or more of these factors.

Another complication of real instruments is that few scales are entirely Pythagorean. In the tenor and bass, string lengths are almost always "foreshortened". This means that moving downward in the scale, string lengths do not become as long as they ought to following Pythagorean proportions. The difference in tension due to foreshortening, or for that matter, any non-pythagorean change in length between pitches, is also very easy to calculate using the semitone approach. In this case, two factors change, l & p:

(Eq.7)

$$\Delta \mathbf{T} = (\Delta \mathbf{l} + \Delta \mathbf{p}) * 2$$

For example, take a string with a length of 1380mm, sounding the F. The string sounding the note E should be 1462mm long, but on a certain instrument we find that it is only 1420mm. The difference between these two numbers in semitones is 0.5; thus the length has gone up by only 1/2 semitone, but pitch has dropped one full semitone, leaving a total change of -1/2 semitone. Tension will drop therefore drop 1 semitone. If the tension of F was 28kg, the reverse semitone formula gives us a value for E of 26,4kg. Once again, this is exactly the result we would get if we actually calculated the tension in the normal way.

Yet another complication of real instruments is that different types of wire are used. The bass is always strung in either yellow or red brass. Naturally these alloys have a different density than iron, both being more dense or heavier, and thus both will produce lower pitches if all other factors are kept constant. John Koster quotes a late 18th century description of this phenomenon, written by one John Dovaston, a gentleman scientist, philosopher, and amateur instrument maker. Dovaston was attempting to explain the reasoning behind Broadwood's adoption of a separate bass bridge with a shorter scale length for the brass strings. Dovaston said, "In the stringing of Harpsichords & piano fortes two sorts of metal are made us of, brass & steel. The reason is that the tone produced by a steel wire of a given length & thickness is higher in its pitch than that of brass wire. . .these different property (sic) of the two metals render brass wire more proper for the lower notes. . .and steel more proper for the higher ones." In all fairness to Dovaston, he also makes mention of steel's superior hardness, which is the real reason why steel, or iron, is used on the longer-scaled treble strings. But suppose we wanted to follow Dovaston's logic and calculate the real difference in pitch of the two metals; one way of course is to arrange Taylor's formula so that tension remains constant and then calculate the difference in pitch with the different densities. But a much faster and easier way becomes apparent if we rearrange the semitone version. Eliminating all factors which remain constant we see that:

(Eq.8)

$$\Delta T=\Delta p*2+\Delta w$$

and since tension remains constant, ΔT is zero, thus

$$\Delta p*2+\Delta w=0$$

or

$$\Delta p*2=-\Delta w$$

and finally

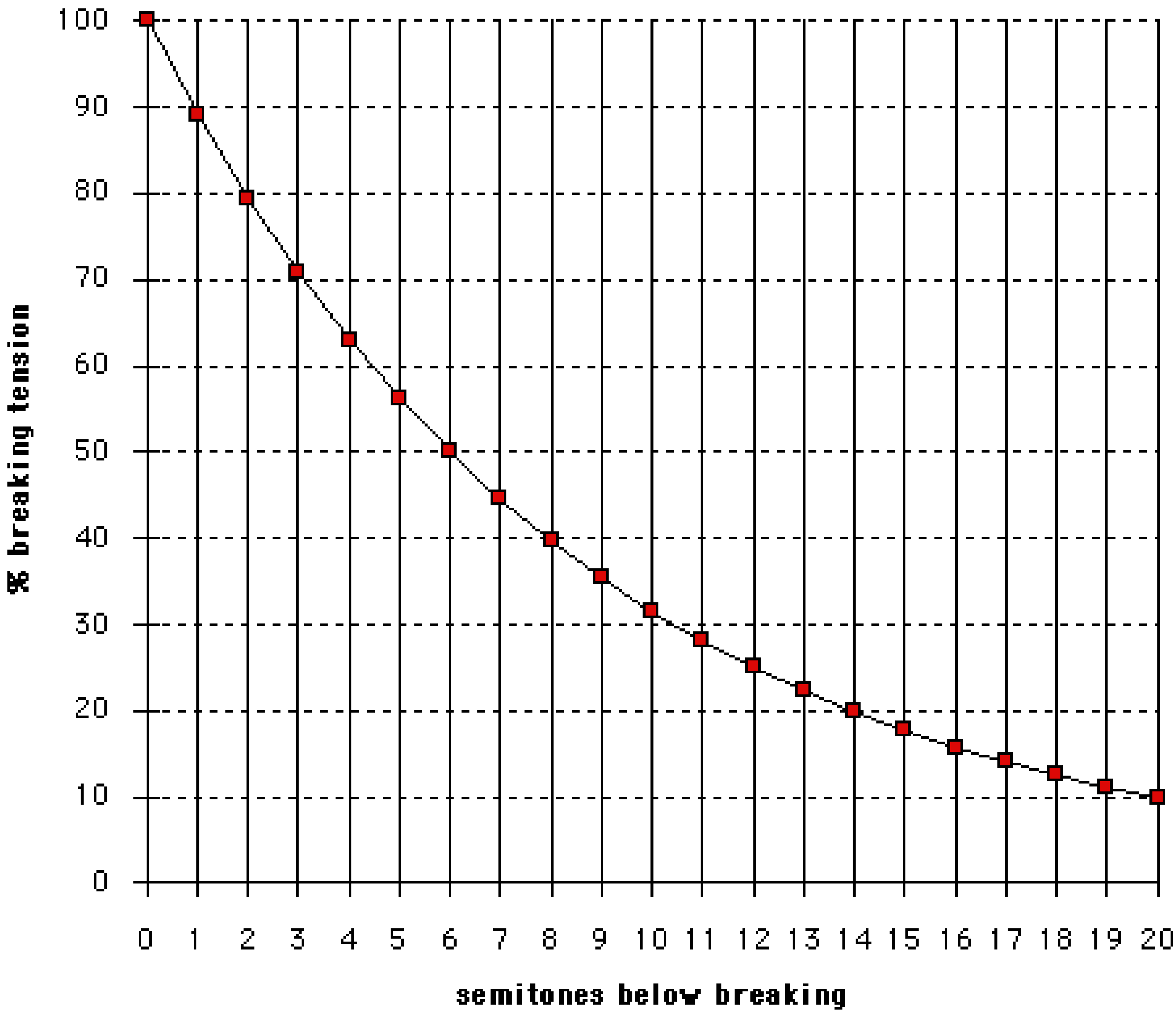
(Eq.9)

$$\Delta p=-\Delta w/2$$

meaning any change in pitch is the inverse of half of any change in density. Malcolm Rose gives densities for his iron and brass of 7769 and 8536kg/m^3, respectively. The difference in semitones between these two values is 1.63; the difference in *pitch*, therefore, would be -0.81 semitones. Thus brass wire of the same diameter and length under the same tension will produce a tone 0.8 semitone lower in pitch than iron wire. When we examine piano and harpsichord scales, we find that brass wire is used on scales anywhere from 2 to 5 semitones shorter than iron scales. Thus the difference in density alone is not enough to account for the use of brass in the bass, as Dovaston vaguely implies.

From these few examples, we have seen how the use of a proportional version of Taylor's formula greatly facilitates the calculation of tension when changing any factor or combination of factors, and how changes in other variables can be deduced when tension is kept constant. But tension, or absolute load in kilograms, is not really the most important factor to analyze either as a builder or researcher. Much more important is the amount of tension on each string in proportion to the maximum tension it can withstand; in other words, how close are the strings to breaking? This factor is called stress. Technically speaking, stress is an absolute measure of the load or force upon a given area, and is measured in Newtons per square millimeter or Megapascals. But I shall use the word in a purely proportional sense. For example, if a certain string will break at a tension of 100 kilograms and it is subjected to a tension of 75 kilograms, the string is said to be "stressed" to 75% of its breaking tension. Stress can also be quickly and easily reckoned with a proportional approach based on semitones, and when we do so, we discover a very efficient and powerful analysis tool.

The first step is to abandon the use of percent to describe stress. Let us instead represent stress as the number of semitones in pitch a given string can be tuned upward before it breaks:



This relationship between semitones pitch and percent breaking tension remains constant regardless of diameter, length, density or the absolute breaking strength. Not only does the use of semitones better describe the underlying relationships, but it is also eminently practical; no player will ever ring you up to ask you if it is safe to increase the stress level of the wire on his instrument by 3%, but he may well ask you if he can safely tune his piano up from 430 to 440, or 1/2 semitone higher.

The next step is to realize that once we describe stress as the pitch of the string relative to its maximum possible pitch, any change in stress becomes equal to any change in pitch. Or:

(Eq.10)

$$\Delta \mathbf{S} = \Delta \mathbf{p}$$

And since we have seen that p and l have the same function in relation to tension, it follows that:

(Eq.11)

$$\Delta S=\Delta l$$

And since we're working with proportional units which are cumulative:

(Eq.12)

$$\Delta S=\Delta l+\Delta p$$

But what about w and d? The fact that different materials have different strengths has nothing to do with their densities. In fact, materials with the same density can be harder or softer drawn. Therefore we exclude w completely because it is not applicable. The question of d is a bit more complicated.

As we have seen, when diameter increases, tension also increases. In standard reference works on musical acoustics and physics, one often reads that the use of different diameters makes no difference in stress, because the increase in *tension* with a thicker string is exactly balanced by the increased *strength* of the thicker string; even though tension goes up, stress remains constant. A practical result of this would be that different diameters of the same type of wire would all break at the same pitch when tested on a monochord. Thus d should also be excluded from our stress equation, and we stay with the simple relationship of Eq. 12:

$$\Delta S=\Delta l+\Delta p$$

This is exactly the sort of reasoning which underlies the often encountered comparisons of overall scale length to overall pitch level. It is assumed that instruments are always strung so as to produce stress levels which are very close to breaking. Thus if stress is to remain more or less constant, changes in length and pitch must always exactly balance each other. Since this is the case on a Pythagorean scale, we can also deduce that stress remains constant on a Pythagorean scale, even if different diameters are used. And since stress on any given Pythagorean scale is the same everywhere, we can compare the stress/tension levels of different overall scale lengths by simply comparing the length of one common reference note from both scales.

Unfortunately, reality is more complex. As Martha Goodway has told us, iron, steel, and brass all become harder and stronger when they are reshaped at a cold temperature, including when they are drawn into wire. Therefore stress *does not* remain constant when different diameters are used for the same string length and pitch, or on scales which are Pythagorean: with each successive change to a smaller diameter, *both* tension *and* stress go down, though the change in stress is always a fraction of the change in tension. The rate at which stress will drop is dependent upon the rate at which the wire strength increases, which in turn depends upon several factors, including the type of material, the amount of diameter reduction of diameter with each drawing, and the speed at which the wire is drawn through the die.

Admittedly, including the effects of tensile strength pick-up in scale analysis seems a daunting task. To get a value for the tensile pick-up rate of a certain type of wire, we must test two different diameters. But which two? With old wire, we have no choice; we are forced to test the diameters of the surviving samples, and we can consider ourselves lucky if we have even two samples. Extrapolating values for <u>all</u> of the diameters used on an instrument would also seem difficult. The situation is even more complicated by the variety of diameters encompassed by different interpretations of old gauge systems. However, using a semitone based system, these problems become manageable.

Tensile strength pick-up is the direct result of the physical act of reducing the diameter of wire, and therefore, there is a correlation between the amount of reduction and the amount of increased strength. Tensile pick-up can therefore be expressed as the increase in pitch capability, in semitones, for a given amount of diameter reduction, say, one semitone. For example, a certain type of wire is tested a monochord and we find that a diameter of 0.90mm breaks at the pitch of c#, while a diameter of 0.75mm breaks at the pitch of d. The thinner wire is thus one semitone stronger than the thicker. The proportional difference in diameter is about 3.5 semitones; therefore, if we divide 1 by 3.5, we see that for every decrease in diameter of 1 semitone, strength goes up 0.28 semitones, or slightly more than 1/4 semitone. The reverse is also true; for every *increase* in diameter of one semitone, the wire gets *weaker* by 1/4 semitone. To extrapolate the strengths for all the other diameters we might use on an instrument, we simply convert the differences in diameter into semitones and then multiply by the strength factor. For example, 0.60mm is exactly 7 semitones smaller than 0.90mm. With our hypothetical pick-up rate of 0.28 semitones, the 0.60 diameter wire would be about two semitones stronger than 0.9, since 7 times 0.28 is 2. If we were to actually test the 0.60mm wire on a monochord, we would expect it to break at d#, or 2 semitones higher than the original 0.90mm string. Using such a system, we eliminate the need to test all the diameters which will be used on an instrument. We can also easily construct hypothetical strength tables for all sizes of wire based on tests of historical samples. Here we can see the tensile pick-up rates for various historical and modern wires expressed in semitones:

Original source	Reporting source	Material	pick-up rate s.t./s.t.
Coulomb, Torsion Mémoire (1784)	Goodway & Odell, "Metallurgy . . ."	brass	0.20
Macolm Rose	published specification	brass	0.15
Macolm Rose	published specification	red brass	0.15
Coulomb, Torsion Mémoire (1784)	Goodway & Odell, "Metallurgy . . ."	iron	0.14
Hofmann f.p. (c.1790), old strings	Latcham/Huber test	iron (Nürnberg?)	0.39
Annales d. Arts & Manufactures (1812)	Gug, Musique.Image.Instr. #1 1995	iron Nürnberg #2 & 3	0.27
Annales d. Arts & Manufactures (1812)	Gug, Musique.Image.Instr. #1 1995	iron Pleyel #2 & 3	0.51
Walter SAM 539, old strings	Huber, "Saitendrahtsysteme..."	iron (Nürnberg?)	0.28
Graf SAM 570, old strings	Latcham/Huber test	iron (Miller?)	0.39
Graf #1594, old strings	Edward Swenson	iron	0.50
Average "realistic" strengths (1916)	Wolfendon, "..Art of P.F. Conts." (1916)	steel #13 & 15	0.25
Macolm Rose	published specification	A iron	0.28
Macolm Rose	published specification	B iron	0.29
Macolm Rose	published specification	C iron	0.31
modern treble wire	Conklin, J. Acoust. Soc. Am.100 (Sept. 96)	steel	0.24
modern bass core wire	Conklin, J. Acoust. Soc. Am.100 (Sept. 96)	steel	0.18
		Iron average	0.32

To factor tensile pick-up into our semitone stress formula, we have to realize that it becomes a modifier as well as an enabler for the diameter factor. We excluded d based on the old assumption that it made no difference, but now we see that it *does* have an effect, but not at its full value. We can factor it in as such:

(Eq.13)

$$\Delta S=\Delta p+\Delta l+(\Delta d*r)$$

when r is our pick-up rate. When d goes up, that is when larger diameters are used, S also goes up, because the maximum pitch the wire can withstand goes down.

While tensile pick-up may seem to be a marginal factor, it can actually produce quite significant effects. We shall soon see how it changes the stress on real instruments. But for the moment, I'd like to illustrate the degree to which the inclusion of tensile pick-up can alter the result of a fairly simple comparison. In his book on Ruckers, Grant O'Brien extrapolates the pitch level of Ruckers instruments through a comparison with instruments of Taskin, using the following assumptions:

1. Ruckers and Taskin used wire of the same overall strength.
2. Ruckers and Taskin chose scalings which would produce the same critical stress level in the wire.
3. The difference in scale length is thus equal to the difference in pitch level.

O'Brien has essentially stated our basic stress formula without tensile pick-up, Eq. 12:

$$\Delta S = \Delta p + \Delta l$$

Since stress and the maximum possible tension (which ultimately determines stress) are both to remain constant, S=0 and any change in l must be compensated by a change in p. In other words;

(Eq.14)

$$\Delta p + \Delta l = 0$$

O'Brien begins with a Taskin pitch level of approximately 409Hz. Taskin's c^2 iron scale is 364mm, and Ruckers' average c^2 is 355mm. Ruckers' scale is 0,4 semitones shorter, and therefore Ruckers' pitch must have been about 0,4 semitones higher, or 418Hz.

However, when we include the effects of tensile pick-up, we arrive at a completely different result. If stress is to remain constant, than any change in p must balance out changes in both l *and* in d*r. Or:

(Eq. 15)

$$\Delta p + \Delta l + (d*r) = 0$$

O'Brien estimates the diameter for c^3 on a Ruckers as 0.31mm; for an English or French 18th century instrument he estimates 0.21mm for the same note. Expressed in semitones, Ruckers is more heavily strung by 6.7 semitones. Using the lowest historical tensile pick-up rate we have seen, that of Coulomb's wire data, we see that the thicker Ruckers strings would be 0.9 semitones weaker. Expressing everything in semitones:

(Eq.16)

$$\Delta \mathbf{d*r=(6.7*0.14)=0.9}$$

and

$$\Delta \mathbf{l=-0.4}$$

thus

$$\Delta \mathbf{p=-0.5}$$

The extrapolated Ruckers pitch would actually be 1/2 semitone *lower* than Taskin, or about 397Hz

Using the average pick-up rate of 0.32 semitones, the situation becomes even worse:

(Eq.17)

$$\Delta \mathbf{d*r=(6.7*0.32)=2.1}$$

and

$$\Delta \mathbf{l=-0.4}$$

thus

$$\Delta \mathbf{p=-1.7}$$

Now the extrapolated Ruckers pitch would have to drop even lower yet to a level of 371Hz. Notice that in order to arrive at this conclusion, we don't need to know anything about the absolute strengths of the wire used by Ruckers or Taskin. We actually do not even need to know what Taskin's pitch level was. All we need is the scale lengths, approximate diameters, and the knowledge that tensile pick-up has always been present in wire. This is enough to perform the calculation in semitones, which gives us a proportion level between the two pitches. As far as I know, there is no current theory in musicology or organology that suggests that 16th/17th century Flemish pitch was so much lower than 18th century French pitch. Therefore, one of the basic assumptions upon which the entire comparison is based must be wrong; either iron wire was being made in different strengths, or not all harpsichord makers adhered to the principle of maximum practical stress.

An interesting clue to resolving this issue becomes apparent if we analyze historical wire data using a semitone system. Strength values can be fitted to different pick-up rates to see if (1) data from different wire samples fit any sort of overall believable pick-up scheme, and (2) which pick-up rate best fits them all.

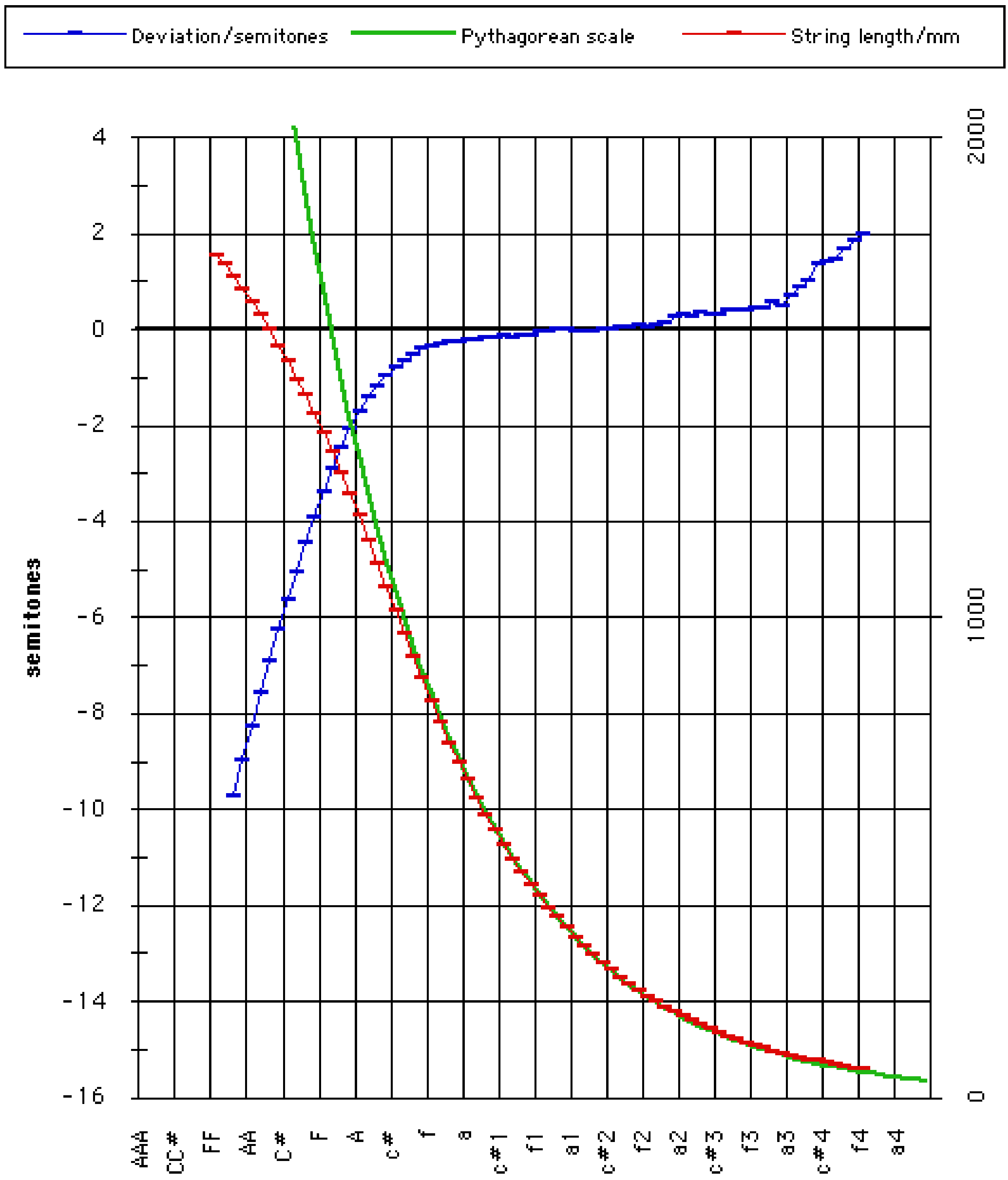
wire	ø mm	strength Mpa	kg	∆ strength in semitones compared with a pick-up rate of: 0.14	0.18	0.32
1733 Blanchet*	0.23	1048	4.4	0.0	0.0	0.0
Coulomb#	0.25	965	4.8	-0.5	-0.4	-0.2
1782 Shudi*	0.29	993	6.7	0.2	0.4	0.9
1732 Vater*	0.33	938	8.2	0.0	0.3	1.1
1730 Willbrook*	0.37	890	9.8	-0.2	0.2	1.3
Mersenne#	0.38	821	9.5	-0.8	-0.5	0.8
Coulomb#	0.50	800	16.0	-0.4	0.2	2.3

*source: Goodway & Odell tests, Table 9
#source: 17th- & 18th century tests, Goodway & Odell Table 5
strength figures from Goodway & Odell
pick-up rates calculated by P.P.

This table shows that if we adopt Coulomb's pick-up rate, the relative overall strength in semitones for all the samples remains consistent within 1 semitone, while the average rate produces an increasing overall strength value in larger diameters, or negative tensile pick-up. The best-fit is achieved with a value slightly higher than Coulomb. This means it is probably true that various types of historical wire was equally strong, at least in the 17th and early 18th centuries, and that Coulomb's rate is more or less believable. However, as we have seen through test of late 18th, 19th, and 20th century wire, Coulomb's rate is much too low for later wire types. This means we have to use different criteria for analyzing pianos than for harpsichords. It also indicates that advances in wire making had probably already begun in the late 18th century, instead of after the turn of the 19th century, as is often assumed.

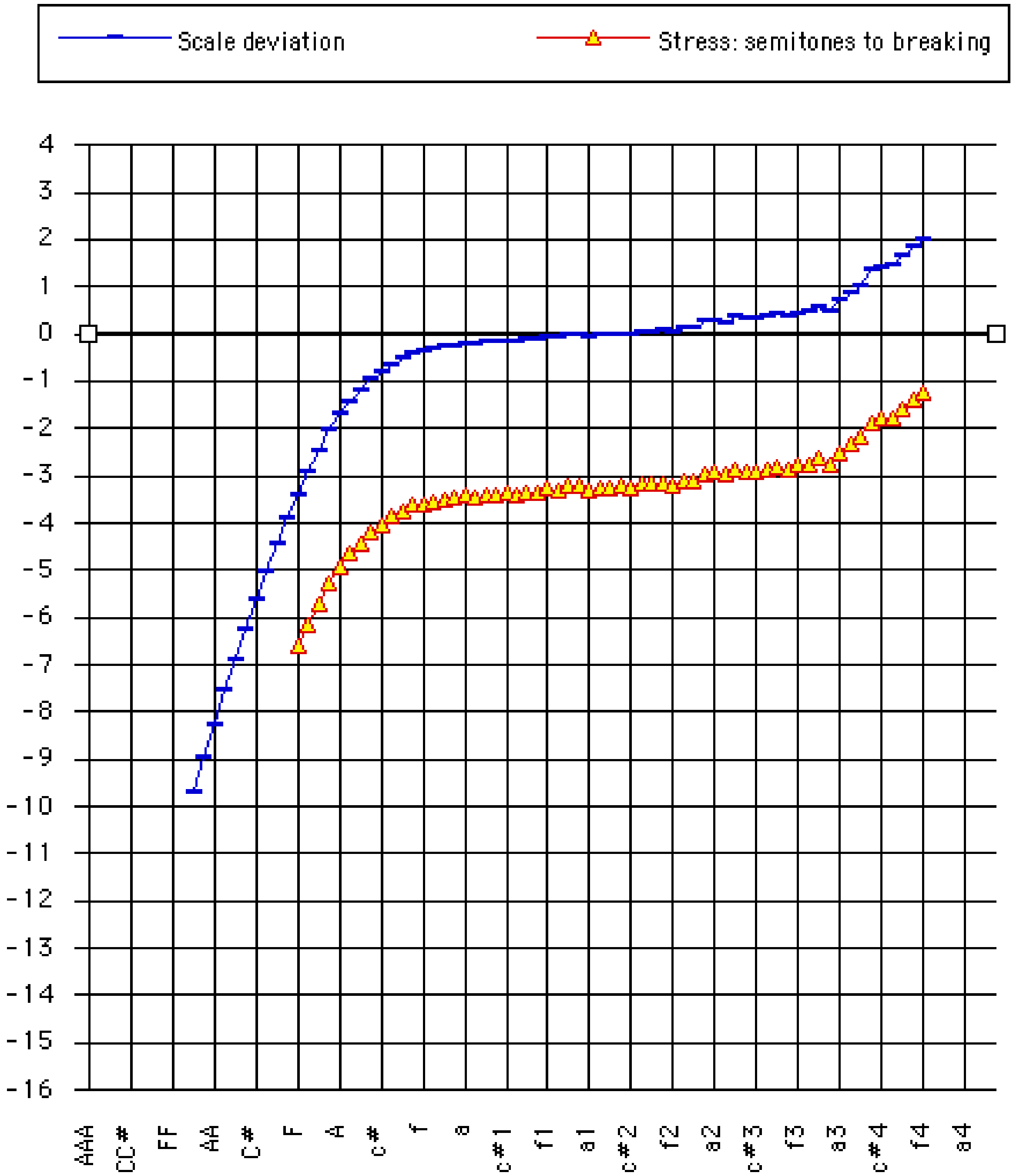
So far we have seen how the use of a semitone-based system simplifies the calculation of tension and stress. But what is perhaps *the* most important advantage becomes apparent only when we graphically express the effects of scale upon stress. If we return for the moment to the simple assumption that tensile pick-up does not exist, we saw that when string lengths are Pythagorean, stress remains constant because changes in pitch are always exactly balanced by changes in length. However, if there is any imbalance between pitch and length, stress will change by exactly this amount. It follows then that any deviation from a Pythagorean scale, in semitones length, will exactly equal any changes in stress, in semitones below breaking. Graphically, if we use the uniform stress of a strict Pythagorean curve as a zero reference line and plot any deviation in string length away from this line, this will show us exactly how the stress upon the strings is changing.

To see how this works, let us examine the scale of a real instrument, the 6 octave Brodmann piano in the Paris conservatory:



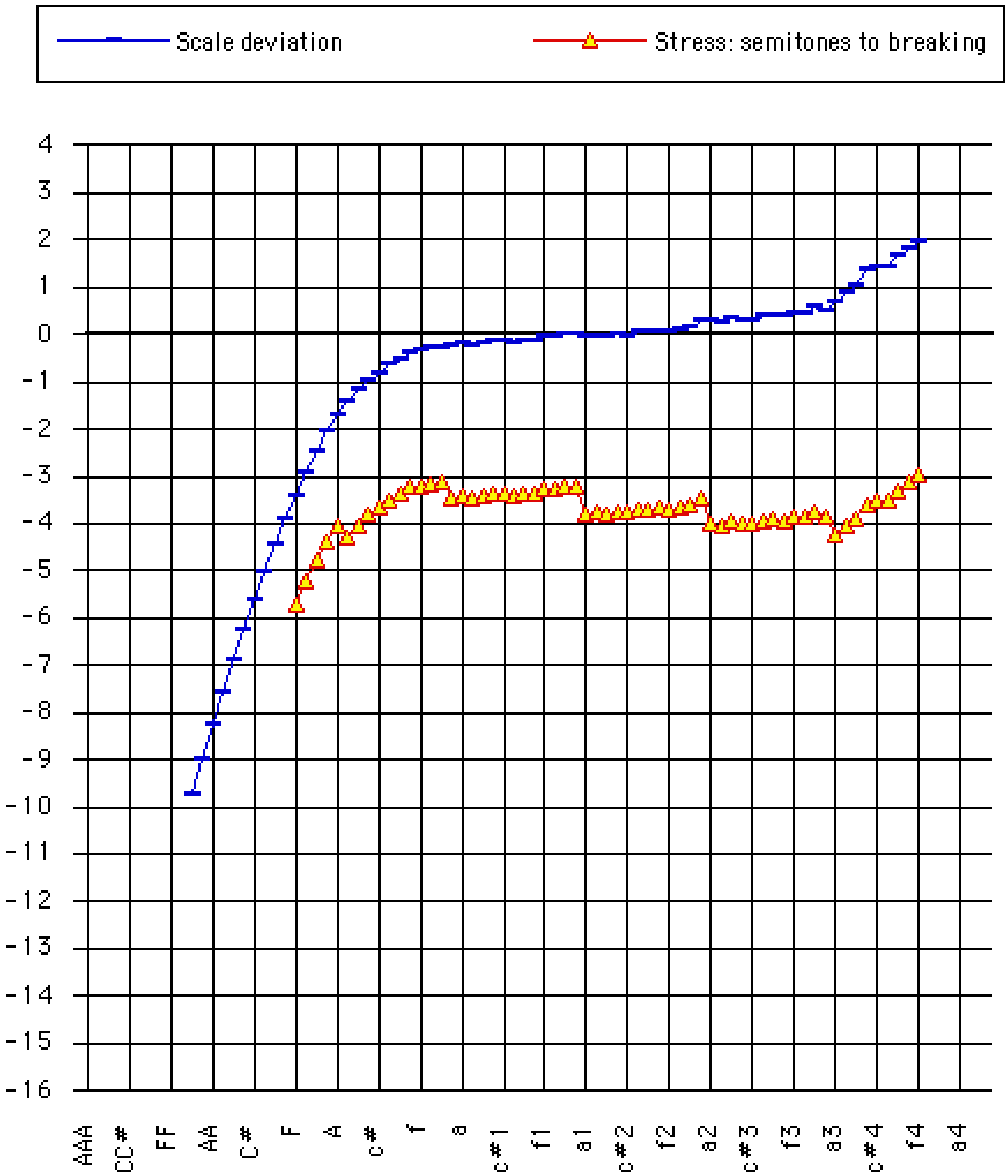
Looking at the actual string lengths against a Pythagorean curve, we see that the scale follows the theoretical curve quite closely for most of the upper part of the instrument. In the tenor, foreshortening cause the strings to become progressively shorter until finally, at the bottom note, FF, they are about half as long as they ought to be. If we look very closely, we can also see the the highest treble strings are also slightly too long. These departures from the theoretical string lengths can also be graphed as semitones deviation from their theoretical lengths, as I have done. The thick “0” line represents a Pythagorean progression; in other words, if the scale were Pythagorean, the values for each string would fall on this line.

Now let’s put some strings on this instrument and see what happens. To begin with, we’ll keep to the idea that tensile pick-up does not exist and that strings of all diameters have the same proportional strength. We’ll take the

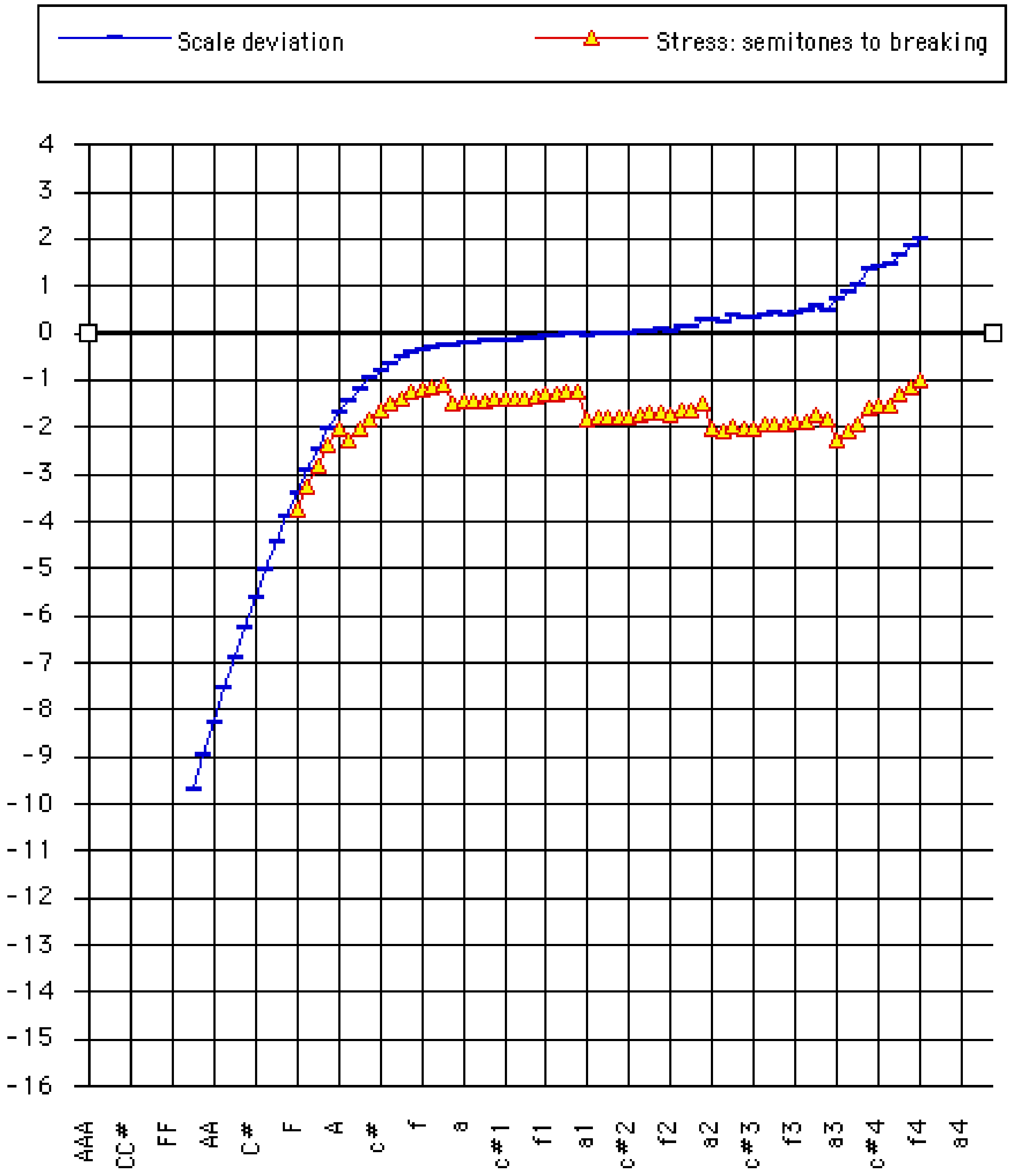


diameter used at middle c, 0,56mm and give it the same breaking tension as Malcolm Rose B iron, 23.4 kg, which is also a very believable value based on historical samples. Using this value, the breaking tension for all other diameters is extrapolated proportionally. Graphing the stress level of all the iron strings, we see that exactly as predicted, the curve of stress level precisely follows the curve of length deviation from Pythagorean. But for most of the treble, the strings are stressed at a level of 3 semitones below breaking, not exactly what one would call “critically stressed”. In the low tenor, the stress drops to a miserable 6 1/2 semitones below breaking, or less than 50% of the breaking tension. However, a weaker wire could not be used, because the top strings are already approaching the maximum limit of 1 semitone below breaking, or 90% of breaking tension, the point at which most wire types enter the plastic deformation zone.

Now let us see what happens when we factor in tensile pick-up. The thickest iron string has a diameter of 0.66mm, which is 2.8 semitones thicker than our reference diameter of 0.56mm. Using the average pick-up rate of 0.32 semitones strength change per semitone diameter change, we would expect the bottom iron strings to become almost one semitone weaker, since 2.8 times 0.32 is almost 1. This means the tenor strings should get one semitone closer to breaking. Our highest treble string has a diameter of 0.41mm, which is 5.4 semitones smaller than our reference diameter. These strings would become 1.7 semitones stronger, and thus drop almost 2 semitones below their current stress level.If we construct a breaking tension table based on our reference diameter and tension factoring in the tensile pick-up rate and then recalculate the stress level for every note, we see that our predictions are exactly right. The overall stress has become much more regular, rising in the tenor and dropping in the treble. But even with the stair-steps caused by tensile pick-up, we can still see that the basic stress curve follows the general shape of the scaling deviation curve.



Our overall stress level is still rather low, though, being nowhere any higher than 3 semitones below breaking. Again, not exactly what one would call ‘critically stressed”. If we accept the idea that critically stressed strings *are* important for good sound (i.e., the idea that strings sound “better” when close to their breaking tensions, regardless of how high or low that tension may be in a n absolute sense), we might want to raise the overall stress level by about 2 semitones. How would we do this?



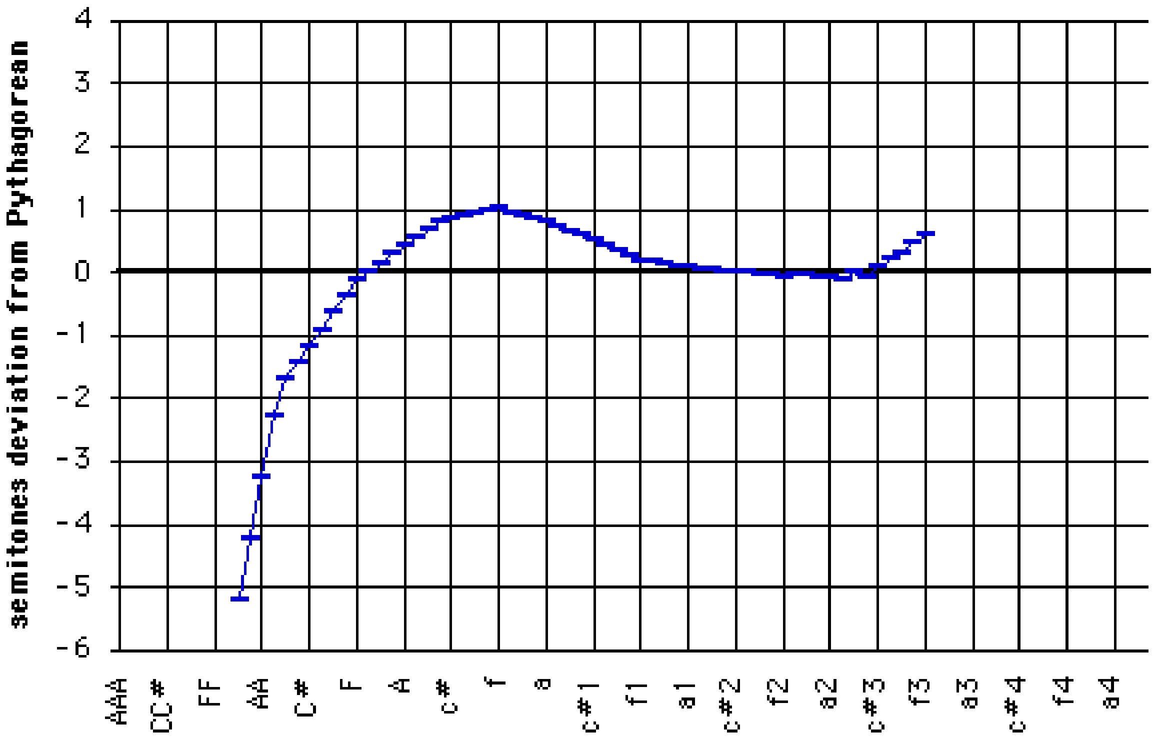
When we converted stress from percentage to semitones, we saw that the proportion between changes in tension and changes in stress was a factor of 2. Therefore, if we want to change the maximum strength of the wire, or the maximum pitch which it can withstand, so as to bring it 2 semitones closer to breaking, we have to lower the breaking tension by 4 semitones. If we take our original tension of 23.4kg for 0.56mm wire and plug it into our semitone formula to calculate 4 semitones less tension, we get a value of 18.6kg. If we now recalculate a breaking-tension table for all the diameters and once again recalculate the stress for the whole instrument, we get exactly the stress levels we predicted: everything has risen by exactly 2 semitones. Thus if we wanted to string this instrument with wire that was truly critically stressed, we would have to seek a type of wire that had tension figures in kilograms about 4 semitones less than Malcolm Rose B, or when strung on a monochord, wire which breaks at pitch levels 2 semitones below Rose B. We wouldn't have to seek far, because this is precisely the strength of Malcolm Rose A wire.

Hopefully, some of you are now beginning the see the beauty of working with semitone units of proportionality. With the aid of one simple formula, one can move quickly and easily through the many complicated factors of scaling and stringing. Once a single graph is generated, using either hypothetical "historical values", values based on extant wire samples, or the real values of currently available wire, one can deduce the effects of all manner of changes: heavier or lighter stringing, stronger or weaker wire, higher or lower pitch levels, and greater or lessor tensile pick-up rates. In fact, one can get a pretty complete picture of how the wire will be stressed on a given instrument merely by looking at the shape of the scale expressed in semitones deviation from Pythagorean. That is why I like to refer to the semitone system as the "unified field theory" for the problems of stringing, scaling, and pitch.

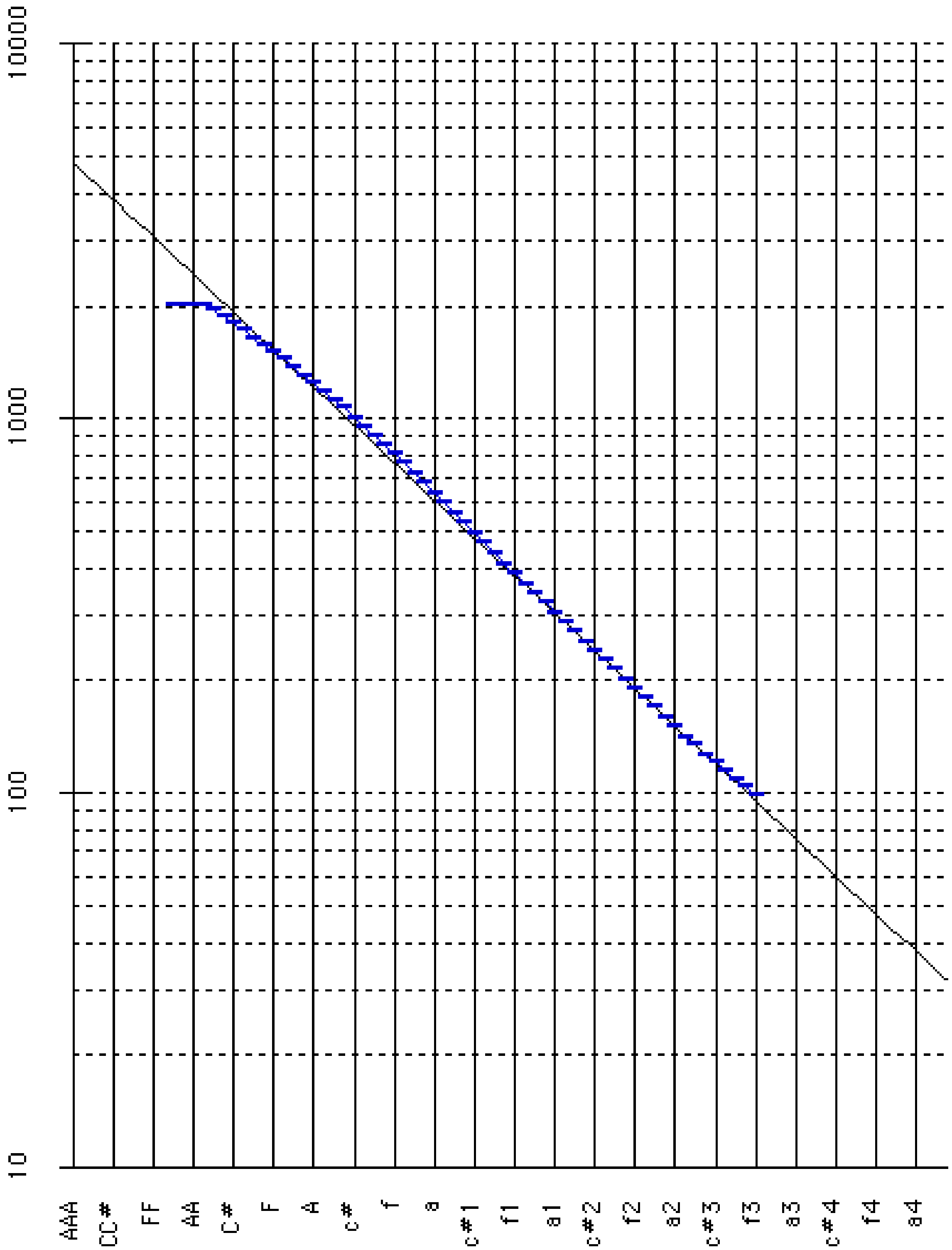
Now that I have explained the basic theory behind the semitone system, I'd like to further demonstrate its flexibility by applying it to a number of recent organological problems. To begin with, I'd like to focus on a what I consider to be a major problem in the current state of scale research and analysis: the overemphasis on the importance and usefulness of c^2 string length. Consider, for example, the idea that there are certain scale lengths which are suitable for brass and iron wire, an idea often encountered in the literature, and that we can determine the appropriate material for a given scale by citing the length of its c^2 string. An extension of this idea is that the best cross-over point between a brass bass and iron treble can be found by converting the string lengths to their c^2 equivalents and noting when they exceed the brass scale length. To illustrate the problem with this logic, let us adapt Mersenne's strength for 0.38mm brass wire and a probable hypothetical pick-up rate of 0.2 semitones. With a pitch level of a=440, we can calculate that with a diameter of 0.20mm, a diameter we might use in the treble of an Italian harpsichord, this brass wire would break with a c^2 length of 334mm. If we increase the diameter to 0.65mm, a common diameter for the highest brass strings in fortepianos, exactly the same brass wire would break at a c^2 of 264mm. This is a proportional difference of 4.1 semitones. Which number shall we choose to represent our Brass Scale length? Since iron seems to exhibit even greater degrees of tensile pick-up, the problem becomes even worse when trying to establish an "Iron Scale" length. This is another aspect of the same problem we saw in the Ruckers/Taskin pitch comparison: when we factor in the differences in strength due to tensile pick-up, we see that the use of c^2 string length, or any standard reference length, to determine the pitch or material suitability for a given scale is so imprecise as to be essentially unreliable. It's a pity that this is so, since organology would be much easier were it not true. And there is also a large body of work based upon the idea that c^2 can be used as a standard. But as they say in America, it's time to wake up and smell the coffee: c^2 is just not going to provide us with valid analysis.

The difficulty of establishing a scale length for given material is only half the problem. The other half is the unreliability of c^2 as an indicator of overall scale length.

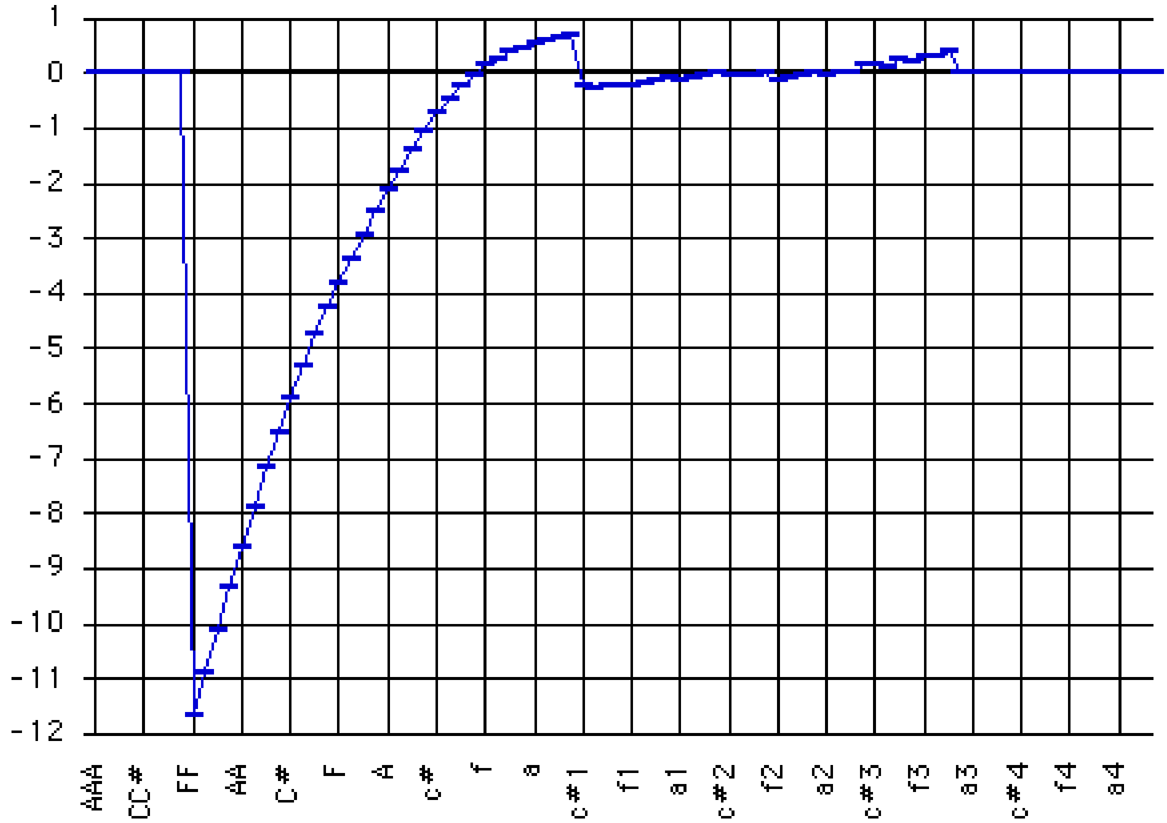
Basically, the idea that any one note can be used to describe an instruments scale is already a risky proposition. This is based on the assumption that there are no unusual departures from the Pythagorean curve, excepting foreshortening, which always lowers stress. The problem is that even when scales are largely Pythagorean, c^2 simply lies in the wrong place: it is too high above the tenor where the strings, either brass or iron, are always the most highly stressed. If the scale of an instrument remains Pythagorean much below middle c, the difference in stress due to tensile pick-up can easily approach 1 semitone, meaning that c^2 gives a scale length which is too optimistic. An extreme scenario would be a scaling which is stretched beyond Pythagorean somewhere below middle c, which is in fact the case on many pianos and even some harpsichords.



This instrument, for example (Leipzig Grassi #89), is stretched by one semitone in the deep tenor, compared to its c^2 length. Such a scaling guarantees that the treble strings will be stressed at levels significantly lower than the maximum possible, a fact which is at odds with the general assumption that instrument scales were always chosen to produce maximum practical stress. This instrument is also a perfect example with which to show the limitation of graphically analyzing scales by plotting string lengths on log paper, an approach often encountered in the literature. Notice how this stretch of one semitone, a critical problem in devising a stringing scheme, is almost impossible to see on such a graph. If we strike a straight line representing a Pythagorean curve, the slight bulge in the tenor becomes apparent, but the visual representation of the data tells us nothing about practical ramifications of this scaling aberration, even if we expand the size of the graphic to almost a full A4 page. This sort of scale deviation, which appears minuscule when so graphed, can have major (i.e., disastrous) consequences in the real world; unfortunately, at least in pianos, such deviations are more the rule than the exception.

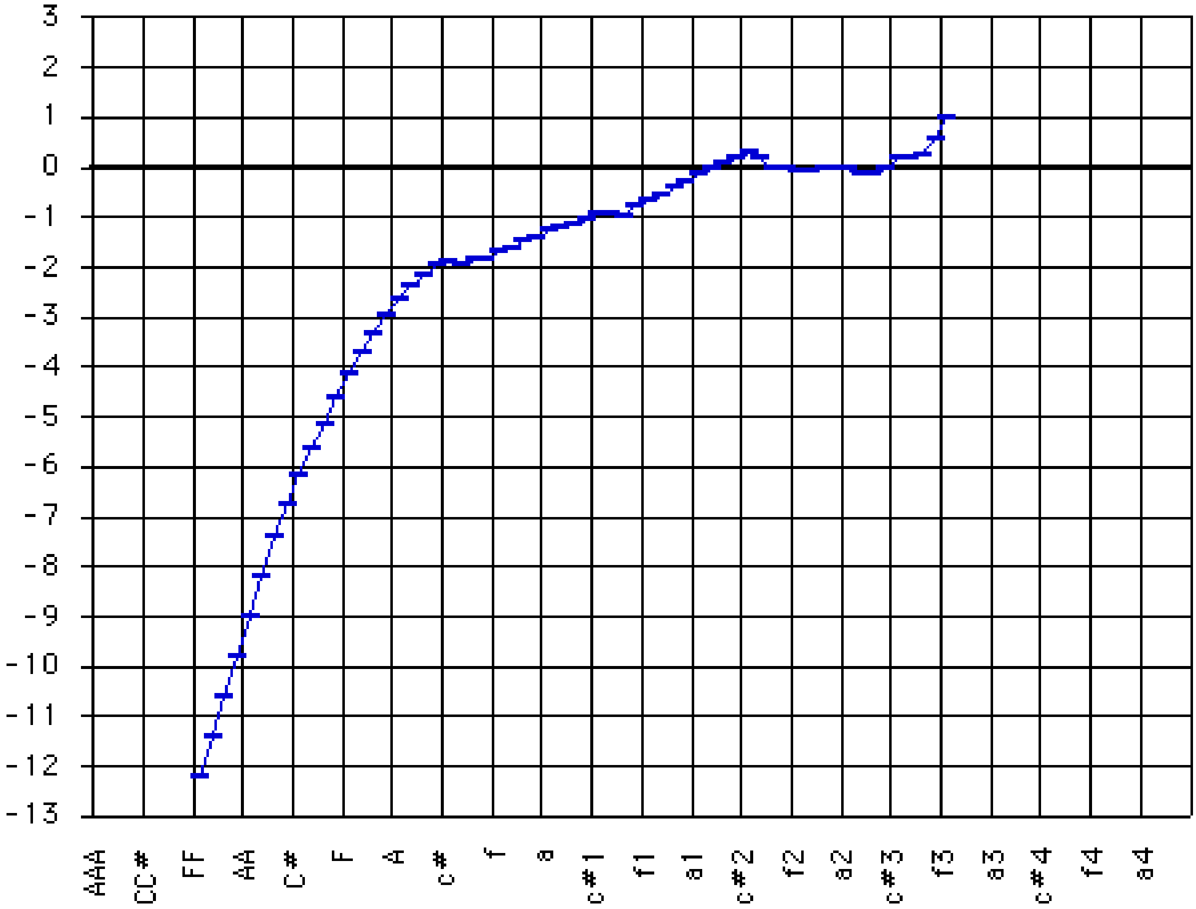


Many pianos of the Viennese school have gap spacers which lie below one choir of strings, and that choir is not used for the production of musical tones. The effect of this “dummy choir” upon the scale depends on whether or not the builder attempted to compensate for the jump. Some builders, such as Stein, Dulcken, Hoffman, and Schantz, simply let their bridge carry on following a Pythagorean shape. For example, on the Smithsonian Dulcken, the spacer is located between the notes b and c1. Going downward from the treble, the spacer lies underneath the set of strings which should have sounded the note b. The actual note b now has the length of a theoretical b flat, meaning that the scale suddenly becomes one semitone longer at this point. If we graph the deviations of the sounding strings only, leaving out the silent choir over the gap spacer, we see that the note b then becomes *the* most critically stressed note on the entire instrument, since it is almost 1 semitone too long compared to the rest of treble scale. It also lies in the tenor region where the wire is always weaker due to the use of larger diameters.



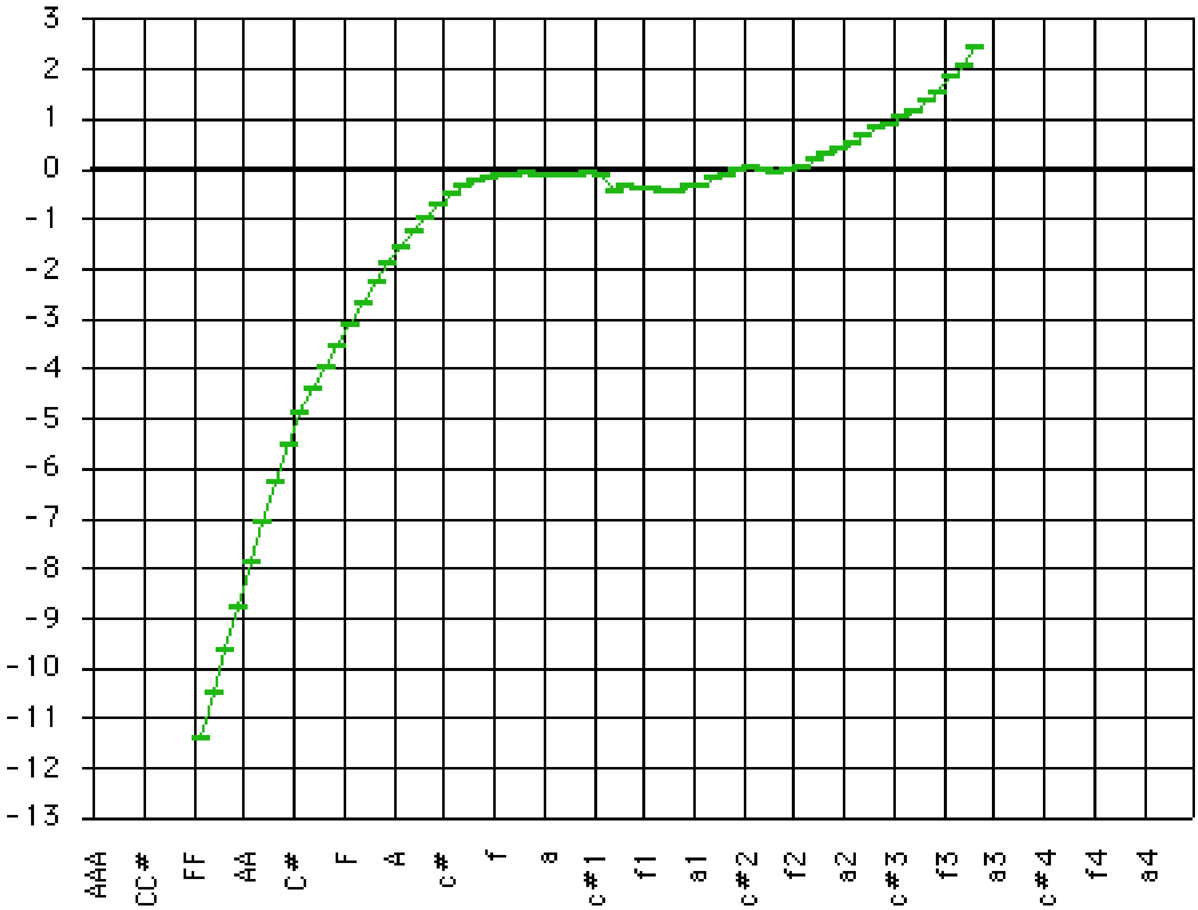
Once again, the length of c^2 gives us a completely inaccurate idea of the critical length of the iron scale on this instrument, since it implies that the overall scale length is one semitone shorter than the practical reality. If we are working with the assumption that iron strings were always stressed very close to their breaking tension and using c^2 as an arbiter of scale length, this error of plus one semitone can make the difference between successful and impractical stringing schemes.

Another interesting example concerns the difference in scaling between the early pianos of Anton Walter and those from around the turn of the 19th century, according to Michael Latcham's proposed dating scheme. The scales of the Mozart and the Eisenstadt instruments are quite similar, with a c^2 of around 300mm. Both instruments have a Pythagorean area of only about 1 octave above c^2, with a high treble stretch of 1 semitone above c3. Below c^2, the scale gradually shortens to about -2 semitones at tenor c, at which point foreshortening begins in earnest. This graph shows the deviation from Pythagorean for the Eisenstadt instrument:

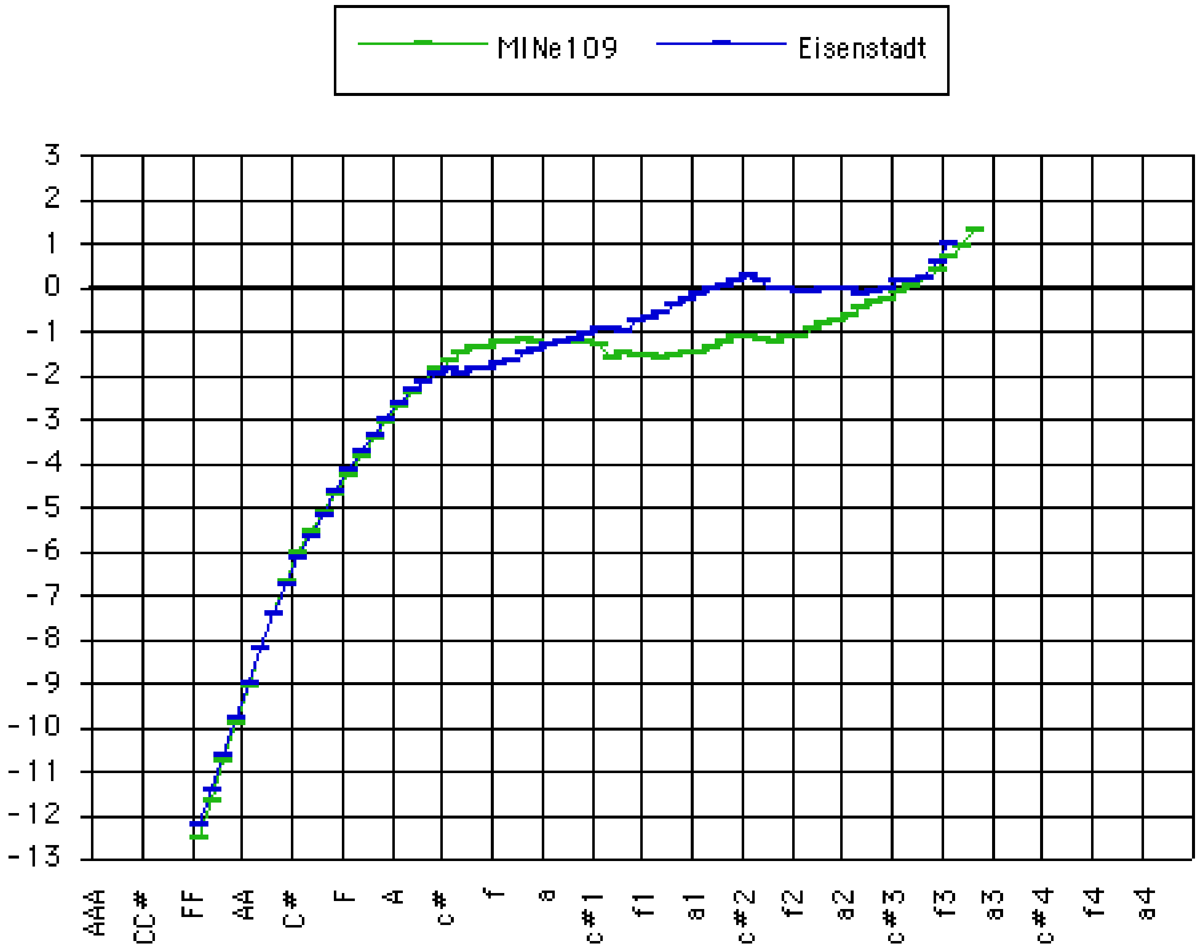


This contrasts clearly with instruments of about 1800, such as the Gamerith, the Kuntshistorishes Museum SAM454, and the Nuremberg MINe109. All of these instruments have a c^2 in the low 280's, a 2 octave essentially-Pythagorean area between f2 and tenor f, at which point foreshortening begins suddenly.

Considering c^2 lengths alone, it would appear as though Walter shortened his iron scaling by about 1 semitone in the later instruments. This fits quite well with the common assumption that early pianos, like harpsichords, were strung so that the wire was very close to its breaking tension, and that scales gradually reduced in length at the end of the 18th century because they were being progressively more heavily strung. Since thicker wire is weaker, builders were forced to shorten their scales until stronger wire became available sometime after the turn of the century, or so the usual story goes. This graph shows the deviation of MINe 109:

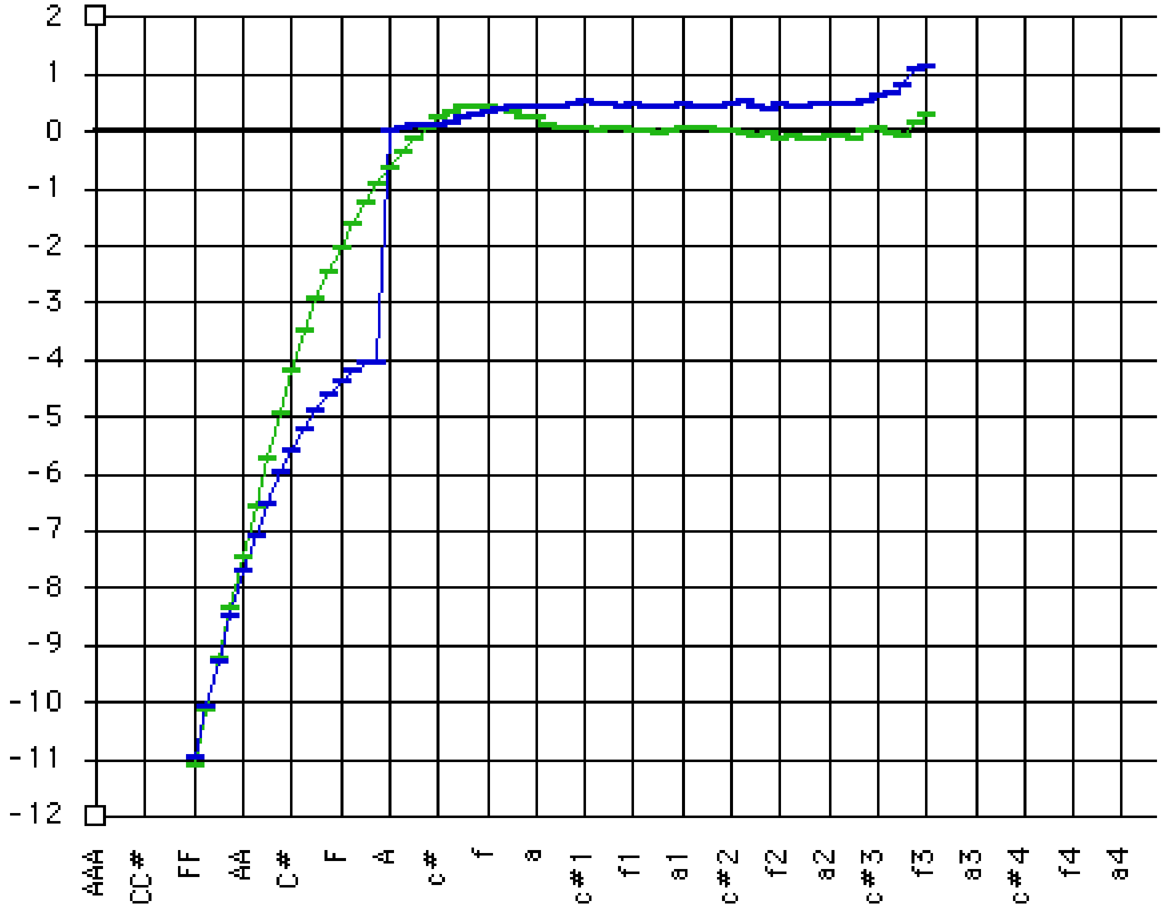


However, a very interesting point becomes clear only when we consider the implications of the lengths of the entire scale of each type of instrument. In the next graph, we see the Eisenstadt and the Nuremberg instruments, both compared to an absolute reference of a Pythagorean curve with a c^2 length of 300mm. Notice that in the two critical stress regions, the tenor just before foreshortening begins and in the highest treble notes, the two instruments are almost exactly the same. In fact, if anything, the *later* instrument is actually slightly *longer* scaled at both points.



If we combine the idea of slightly longer scaling in the most critical regions with the assumption that the later instruments were more heavily strung, and we arrive at an interesting paradox. Assuming that pitch remained constant between 1780 and 1800, there are only two possibilities: (1) if there was indeed no difference in strength in wire during that time period, then the early instruments *must* have been less critically strung in order to allow Walter to string the later instruments more heavily without shortening the scale *in the danger zones*; or (2) if the early instruments *were* strung very close to the breaking points, then there *must* have been an increase in wire strength, for wire which was close to breaking on an Eisenstadt instrument would certainly have broken on a MINe109 instrument with even slightly heavier diameters. Once again the use of c^2 length alone for scale comparison obscures a very important reality, and leads us to conclusions which are contradictory with what was actually possible. Analyzing the entire scale in semitones deviation forces us to think like builders, because it graphically confronts us with the real trouble spots, the *absolute limitations* which could not have been exceeded.

Leaving aside the problem of increased wire diameter, what Walter actually did was to increase his stress safety margin in the area of c^2 by shortening the scale in that region *only*. This is also the region on a Walter piano which is triple strung. According to Robert Brown, both the Mozart and Eisenstadt instruments have curious layouts of nut and tuning pins for the left string of each choir in the treble which strongly suggest that the third string was added at a later date. The usual assumption is that the function of triple stringing in the treble is to increase the volume in this normally weak register. But modern acoustic theory tells us that a third string only increases volume by about 10%. Perhaps the third string was really added to solve the problem of breaking strings, since a third string adds more resistance to upward motion of the hammer. Perhaps adding a third string did not completely resolve this problem, so Walter shortened his scale to reduce stress in this region, which would also make the strings more durable under the force of the hammer blow.



For a final example of the lucidity of the semitone system, I'll return to the beginning of my lecture and John Koster's complicated proof that Broadwood lowered the maximum stress on the brass strings to "4 or 5 semitones" below breaking when he changed to divided bridges. As we have seen, stress is directly proportional to length. Therefore, the same conclusion could have been made simply calculating the difference in the length in semitones of the G# strings on both types of instrument. The chart above shows the comparison. G# on a 1787

Broadwood with a single bridge is 1290mm long, and on an early divided bridge Broadwood, it is 1076mm; this difference is 3.1 semitones. If we begin with the idea that the maximum stress possible on the single bridge instrument was 1 to 2 semitones below breaking, we end up with the same conclusion, only in a much more direct manner.

Unfortunately, my time is more than up. There are many more advantages to using a semitone based system than I have been able to present here today, and other significant problems in organology that can easily be addressed through such an approach. In closing, all I can say is, “try it. . .you’ll like it.”